

Nonuniform power instability and Lyapunov sequences

Ioan-Lucian Popa^c, Traian Ceașu^b, Mihail Megan^{a,b}

^a*Academy of Romanian Scientists, Independenței 54, 050094 Bucharest, Romania*

^b*Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timișoara, V. Pârvan Blv. No. 4, 300223-Timișoara, Romania*

^c*Department of Mathematics, University "1st December 1918" of Alba Iulia, 510009 Alba Iulia, Romania*

Abstract

The aim of this paper is to present necessary and sufficient conditions for nonuniform power instability property of linear discrete-time systems in Banach spaces. A characterization of the nonuniform power instability in terms of Lyapunov sequences is also given.

Keywords: uniform power instability, nonuniform power instability, Lyapunov sequences

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The development of mathematical dynamical systems theory can be viewed as the simultaneous pursuit of two lines of research: on the one hand, the quest for simplicity, comprehensibility, stability; on the other hand, the discovery of complexity, instability, chaos.
Morris W. Hirsch, [4], pag. 26

1. Introduction

The instability problem together with the stability and dichotomy problems became one of special interests in the field of the asymptotic behavior of linear discrete-time systems. In this context, there are different characterizations of instability in the papers due to N. van Minh, F. Răbiger, and R.

Email addresses: `lucian.popa@uab.ro` (Ioan-Lucian Popa), `ceausu@math.uvt.ro` (Traian Ceașu), `megan@math.uvt.ro` (Mihail Megan)

Schnaubelt [10], M. Megan, A. L. Sasu and B. Sasu [8]-[9], R. Naulin and C. J. Vanegas [11], A.L. Sasu [15], V.E. Slyusarchuk [16], B.-G. Wang and Z.-C. Wang [17].

In [15] A.L. Sasu obtained so-called theorems of Perron type for exponential instability of one parameter semigroups. This method is generalized and applied by M. Megan, A. L. Sasu and B. Sasu in [8] for the study of evolution operators and in [9] for the linear skew-product flows. Also in [8] generalizations to the nonuniform case of some results of N. van Minh, F. Rabiger, and R. Schnaubelt [10] are obtained. In [17] B.-G. Wang and Z.-C. Wang characterize instability from the hyperbolic point of view.

The importance of Lyapunov functions (sequences) is well established in the study of linear and nonlinear systems in both continuous and discrete-time. Thus, after the seminar work of A. M. Lyapunov (republished in 1992 [6]) relevant results using the Lyapunov's direct method are presented in the books due to J. LaSalle and S. Lefschetz [5], W. Hahn [2], A. Halanay and D. Wexler [3] for linear discrete systems and in the review of A.A. Martynyuk [7] for nonlinear discrete systems. In [14] (Theorem 23.6) W. Rugh present the existence of a quadratic Lyapunov sequence in order to develop instability criteria for finite-dimensional case. This result it is extended by J.J. DaCunha [1] for the timescale case.

This paper is organized as follows. In Section 2 and Section 3 we focus our attention on some properties of uniform and nonuniform power instability of linear discrete-time systems. Thus, we establish relations between these concepts and we give some theorems of characterizations for these concepts of linear discrete-time systems in Banach spaces. In Section 4 we introduce the notion of Lyapunov sequence and we show how nonuniform power instability can be characterized in terms of Lyapunov sequences.

2. Power instability

Let X be a real or complex Banach space. We consider the linear discrete-time system

$$x_{n+1} = A(n)x_n, \text{ for all } n \in \mathbb{N}, \quad (\mathfrak{A})$$

where $x_n \in X$ and the operators $A(n)$ belong to $\mathcal{B}(X)$, the space of all bounded linear operators on X . Throughout the paper, the norm on X and on $\mathcal{B}(X)$ will be denoted by $\| \cdot \|$.

We recall that if

$$\Delta = \{(m, n) \in \mathbb{N} : m \geq n\}$$

then the solution $x = (x_n)$ of (\mathfrak{A}) is given by $x_m = \mathcal{A}(m, n)x_n$, for all $(m, n) \in \Delta$ where $\mathcal{A} : \Delta \rightarrow \mathcal{B}(X)$ is defined by

$$\mathcal{A}(m, n) = \begin{cases} A(m-1) \cdot \dots \cdot A(n), & m \geq n+1 \\ I, & m = n. \end{cases} \quad (1)$$

Clearly, $\mathcal{A}(m, n)\mathcal{A}(n, p) = \mathcal{A}(m, p)$, for all $(m, n), (n, p) \in \Delta$.

Definition 1. The linear discrete-time system (\mathfrak{A}) is said to be *uniformly power instable* (UPIS) if there are some constants $N \geq 1$ and $r \in (0, 1)$ such that:

$$\|\mathcal{A}(n, p)x\| \leq Nr^{m-n} \|\mathcal{A}(m, p)x\|, \quad (2)$$

for all $(m, n), (n, p) \in \Delta$ and all $x \in X$.

Definition 2. The linear discrete-time system (\mathfrak{A}) is said to be *nonuniformly power instable* (NPIS) if there exist a nondecreasing sequence $N : \mathbb{N} \rightarrow [1, \infty)$ and $r \in (0, 1)$ such that:

$$\|\mathcal{A}(n, p)x\| \leq N(m)r^{m-n} \|\mathcal{A}(m, p)x\|, \quad (3)$$

for all $(m, n), (n, p) \in \Delta$ and all $x \in X$.

Two particular cases of nonuniform power instability are introduced by

Definition 3. The linear discrete-time system (\mathfrak{A}) is said to be

i) *power instable* PIS if there are some constants $N \geq 1$, $r \in (0, 1)$ and $s \geq 1$ such that:

$$\|\mathcal{A}(n, p)x\| \leq Nr^{m-n}s^n \|\mathcal{A}(m, p)x\|, \quad (4)$$

for all $(m, n), (n, p) \in \Delta$ and all $x \in X$.

ii) *strongly power instable* SPIS if there are some constants $N \geq 1$, $r \in (0, 1)$ and $s \in \left[1, \frac{1}{r}\right)$ such that:

$$\|\mathcal{A}(n, p)x\| \leq Nr^{m-n}s^n \|\mathcal{A}(m, p)x\|, \quad (5)$$

for all $(m, n), (n, p) \in \Delta$ and all $x \in X$.

It is obvious that if system (\mathfrak{A}) is UPIS than it is NPIS. The following example shows that the converse implication is not valid.

Example 1. Let $c \in \mathbb{R}_+^*$, $b \geq 2$ and (\mathfrak{A}) be the linear-time system defined for all $n \in \mathbb{N}$ by

$$A_n = c \cdot a_n I, \text{ where } a_n = \begin{cases} b^{-n} & \text{if } n = 2k \\ b^{n+1} & \text{if } n = 2k + 1 \end{cases}.$$

The following statements are true:

- i) (\mathfrak{A}) is not UPIS for all $c \in \mathbb{R}_+^*$;
- ii) if $c > 1$, then (\mathfrak{A}) is NPIS.

Let $(m, n, x) \in \Delta \times X$. According to (1) we have that

$$\mathcal{A}(m, n)x = \begin{cases} c^{m-n} a_{mn} x & m > n \\ x & m = n \end{cases},$$

where

$$a_{mn} = \begin{cases} b^{m-n} & \text{if } m = 2q \text{ and } n = 2p \\ b^m & \text{if } m = 2q \text{ and } n = 2p + 1 \\ b^{-n} & \text{if } m = 2q + 1 \text{ and } n = 2p \\ 1 & \text{if } m = 2q + 1 \text{ and } n = 2p + 1 \end{cases}$$

(i) If we suppose that (\mathfrak{A}) is UPIS then there exist some constants $N \geq 1$ and $r \in (0, 1)$ such that

$$\|x\| \leq N r^{m-n} \|\mathcal{A}(m, n)x\| = N (rc)^{m-n} a_{mn} \|x\|$$

for all $(m, n, x) \in \Delta \times X$, which is equivalent with

$$\left\{ \begin{array}{ll} \left(\frac{1}{rc}\right)^{m-n} b^{-m+n} \leq N & \text{if } m = 2q \text{ and } n = 2p \\ \left(\frac{1}{rc}\right)^{m-n} b^{-m} \leq N & \text{if } m = 2q \text{ and } n = 2p + 1 \\ \left(\frac{1}{rc}\right)^{m-n} b^n \leq N & \text{if } m = 2q + 1 \text{ and } n = 2p \\ \left(\frac{1}{rc}\right)^{m-n} \leq N & \text{if } m = 2q + 1 \text{ and } n = 2p + 1. \end{array} \right.$$

There are two cases that can be considered at this point. If $c \in (0, 1]$ then for $m = 2q + 1$ and $n = 2p + 1 \in \mathbb{N}$ fixed we have that

$$\lim_{q \rightarrow \infty} \left(\frac{1}{rc} \right)^{2q-2p} = \infty. \quad (6)$$

If $c \in (1, \infty)$, $n = 2p$ and $m = n + 1$ it follows that

$$\lim_{p \rightarrow \infty} \left(\frac{1}{rc} \right) b^{2p} = \infty. \quad (7)$$

According to (6) and (7) we can conclude that (\mathfrak{A}) can not be UPIS.

(ii) If $c > 1$ then for $r = \frac{1}{cb}$ and $N(m) = b^m$ the system (\mathfrak{A}) is NPIS.

Remark 1. Previous Example (with $b = 2$) was studied in [13] in order to prove that the concepts of PIS and NPIS are not equivalent. Thus, the system (\mathfrak{A}) is PIS if and only if $c \geq 1$ and SPIS if and only if $c \geq 2$.

Proposition 1. *The linear discrete-time system (\mathfrak{A}) is NPIS if and only if there are two nondecreasing sequences $\varphi, \tau : \mathbb{N} \rightarrow [1, \infty)$ with $\lim_{n \rightarrow \infty} \tau(n) = \infty$ such that*

$$\tau(m - n) \| x \| \leq \varphi(m) \| \mathcal{A}(m, n)x \|, \quad (8)$$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity. It is a simple verification for $\varphi(m) = N(m)$ and $\tau(m) = r^{-m}$ where $r \in (0, 1)$ is given by Definition 2.

Sufficiency. Using the hypothesis $\lim_{n \rightarrow \infty} \tau(n) = \infty$ we have that there exists $c \in \mathbb{N}^*$ with $\tau(c) > 1$. Then for all $(m, n, x) \in \Delta \times X$ there exist two constants $k \in \mathbb{N}$ and $s \in \{0, 1, \dots, c-1\}$ such that $m = n + kc + s$. According

to (8) we have that

$$\begin{aligned}
\| \mathcal{A}(m, n)x \| &\geq \frac{\tau(s)}{\varphi(m)} \| \mathcal{A}(n + kc, n)x \| , \\
\| \mathcal{A}(n + kc, n)x \| &\geq \frac{\tau(c)}{\varphi(n + kc)} \| \mathcal{A}(n + (k - 1)c, n)x \| , \\
\| \mathcal{A}(n + (k - 1)c, n)x \| &\geq \frac{\tau(c)}{\varphi(n + (k - 1)c)} \| \mathcal{A}(n + (k - 2)c, n)x \| , \\
&\vdots , \\
\| \mathcal{A}(n + 2c, n)x \| &\geq \frac{\tau(c)}{\varphi(n + 2c)} \| \mathcal{A}(n + c, n)x \| , \\
\| \mathcal{A}(n + c, n)x \| &\geq \frac{\tau(c)}{\varphi(n + c)} \| x \| ,
\end{aligned}$$

which implies

$$\| \mathcal{A}(m, n)x \| \geq \frac{\tau(s)\tau(c)^k}{\varphi(m) \cdot \varphi(n + kc) \cdot \dots \cdot \varphi(n + c)} \| x \| .$$

Furthermore, since

$$\varphi(m) \geq \varphi(n + kc) \geq \dots \geq \varphi(n + 2c) \geq \varphi(n + c),$$

we obtain

$$\begin{aligned}
\varphi(m)^{k+1} \| \mathcal{A}(m, n)x \| &\geq \tau(0)\tau(c)^k \| x \| \\
&= \tau(0)r^{-(m-n)}r^s \| x \| ,
\end{aligned}$$

where $r = e^{-\frac{\ln \tau(c)}{c}}$. Thus, we can conclude that (\mathfrak{A}) is NPIS. \square

A similar result with Proposition 1 for the concept of UPIS is given by

Corollary 2. *The linear discrete-time system (\mathfrak{A}) is UPIS if and only if there exists a nondecreasing sequence $\varphi : \mathbb{N} \rightarrow \mathbb{R}_+^*$ with $\lim_{n \rightarrow \infty} \varphi(n) = \infty$ such that*

$$\varphi(m) \| x \| \leq \| \mathcal{A}(m + n, n)x \| , \text{ for all } (m, n) \in \mathbb{N}^2 \text{ and all } x \in X.$$

Remark 2. For time-invariant linear systems, (i.e. $A(n) = A$ and $\mathcal{A}(m, n) = A^{m-n}$) we have that $\lim_{n \rightarrow \infty} \|A^n\| = \infty$ is a necessary and sufficient condition for the concept of UPIS.

In the following example we show that for time-varying linear systems the relation

$$\lim_{m \rightarrow \infty} \|\mathcal{A}(m, n)\| = \infty \quad (9)$$

represents just a necessary condition for UPIS.

Example 2. Let (\mathfrak{A}) be the linear-time system given by

$$A(n) = \frac{u(n+1)}{u(n)} I, \quad (10)$$

where $u(n) = n + 2$, for all $n \in \mathbb{N}$. According to (1) we have that

$$\mathcal{A}(m, n)x = \frac{m+2}{n+2}x, \text{ for all } (m, n, x) \in \Delta \times X.$$

Firstly, we observe that $\lim_{m \rightarrow \infty} \|\mathcal{A}(m, n)\| = \infty$. On the other hand, if we suppose that (\mathfrak{A}) is UPIS then there exists $N \geq 1$ and $r \in (0, 1)$ such that

$$r^n \leq N \frac{m+n+2}{n+2} r^m = \frac{N}{n+2} m r^m + N r^m.$$

From here, for fixed $n \in \mathbb{N}$ and $m \rightarrow \infty$ we have that $0 < r^n \leq 0$ which is false.

Note that the property (9) is not valid for the concept of NPIS. This fact is illustrated by

Example 3. Let (\mathfrak{A}) be the discrete-time system given by (10), with $u(n) = e^{-n}$, for all $n \in \mathbb{N}$. Thus, we obtain

$$\mathcal{A}(m, n)x = e^{n-m}x, \text{ for all } (m, n, x) \in \Delta \times X.$$

Obvious, (9) it is false. Furthermore, the system (\mathfrak{A}) is not UPIS. Finally, we remark that for $N(n) = e^{n^2}$ and $r = e^{-2}$ the inequality (3) is satisfied and thus the system (\mathfrak{A}) is NPIS.

We can observe that the system considered in Example 3 is uniformly exponentially stable. For further information about (non)uniform exponential stability concepts see [12]. There are some particular concepts of NPIS for what the property (9) holds (see [13]).

3. Auxiliary results.

Theorem 3. *The linear discrete-time system (\mathfrak{A}) is NPIS if and only if there are some constants $d > 1$, $p \in (0, +\infty)$ and a nondecreasing sequence $\theta : \mathbb{N} \rightarrow [1, +\infty)$ such that*

$$\sum_{k=n}^m d^{p(m-k)} \| \mathcal{A}(k, n)x \| ^p \leq \theta(m)^p \| \mathcal{A}(m, n)x \| ^p, \quad (11)$$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity. Using Definition 2 we have that

$$\begin{aligned} \sum_{k=n}^m d^{p(m-k)} \| \mathcal{A}(k, n)x \| ^p &\leq \sum_{k=n}^m d^{p(m-k)} N(m)^p r^{p(m-k)} \| \mathcal{A}(m, n)x \| ^p \\ &= N(m)^p \| \mathcal{A}(m, n)x \| ^p \sum_{k=n}^m [(rd)^p]^{m-k} \\ &\leq \frac{N(m)^p}{1 - (rd)^p} \| \mathcal{A}(m, n)x \| ^p \end{aligned}$$

for any $d \in \left(1, \frac{1}{r}\right)$ and all $(n, x) \in \mathbb{N} \times X$.

Sufficiency. The inequality (11) implies that

$$d^{p(m-n)} \| x \| ^p \leq \theta(m)^p \| \mathcal{A}(m, n)x \| ^p, \quad \text{for all } (m, n, x) \in \Delta \times X.$$

From this it results that

$$\| x \| \leq \theta(m) \left(\frac{1}{d}\right)^{m-n} \| \mathcal{A}(m, n)x \|$$

and thus the system (\mathfrak{A}) is NPIS. \square

Corollary 4. *The linear discrete-time system (\mathfrak{A}) is UPIS if and only if there are some constants $p \in (0, +\infty)$, $D \geq 1$, $d > 1$ such that*

$$\sum_{k=n}^m d^{p(m-k)} \| \mathcal{A}(k, n)x \| ^p \leq D^p \| \mathcal{A}(m, n)x \| ^p, \quad (12)$$

for all $(m, n, x) \in \Delta \times X$.

Proof. It results as a particular case from Theorem 3. \square

According to the previous theorem we can obtain necessary and sufficient criteria for other (particular) cases of instability. More precisely

Proposition 5. *The linear discrete-time system (\mathfrak{A}) is PIS if and only if there are some constants $p \in (0, +\infty)$, $D \geq 1$, $d > 1$ and $c > 1$ with $c \in (1, d)$ such that*

$$\sum_{k=n}^m d^{p(m-k)} \| \mathcal{A}(k, n)x \| ^p \leq D^p c^{pm} \| \mathcal{A}(m, n)x \| ^p, \quad (13)$$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity. Using Definition 3 (i) we have that $1 < \frac{1}{r} \leq \frac{s}{r}$. Thus, for $d > 1$ with $0 < \frac{s}{rd} < 1$ we have that

$$\begin{aligned} \sum_{k=n}^m d^{p(m-k)} \| \mathcal{A}(k, n)x \| ^p &\leq N^p (rd)^{pm} \| \mathcal{A}(m, n)x \| ^p \sum_{k=n}^m \left[\left(\frac{s}{rd} \right)^p \right]^k \\ &\leq \frac{N^p (rd)^{pm}}{1 - \left(\frac{s}{rd} \right)^p} \| \mathcal{A}(m, n)x \| ^p, \end{aligned}$$

for all $(n, x) \in \mathbb{N} \times X$.

Sufficiency. It is easy to see that, for all $(m, n, x) \in \Delta \times X$ we have the following inequality

$$\| x \| \leq D \left(\frac{c}{d} \right)^{m-n} c^n \| \mathcal{A}(m, n)x \| .$$

Thus, we can conclude that system (\mathfrak{A}) is PIS. \square

Proposition 6. *The linear discrete-time system (\mathfrak{A}) is SPIS if and only if there are some constants $p \in (0, +\infty)$, $D \geq 1$, $d > 1$ and $c > 1$ with $c^2 < d$ such that (13) hold for all $(m, n, x) \in \Delta \times X$.*

Proof. Necessity. Since $0 < r < 1 \leq s < \frac{1}{r}$ we have that $1 \leq \frac{s}{r} < \frac{1}{r^2}$. In the same manner as we proved Proposition 5, if $d > 1$ such that $\frac{s}{r} < d < \frac{1}{r^2}$ and $c = rd > s \geq 1$, then relation (13) it is verified for all $(n, x) \in \mathbb{N} \times X$.

Sufficiency. This follows using similar arguments to those in the proof of Proposition 5. \square

4. Lyapunov sequences

Definition 4. We say that $L : \Delta \times X \rightarrow \mathbb{R}_+$ is a *Lyapunov sequence* for the system (\mathfrak{A}) if there exists a constant $a \in (1, \infty)$ such that

$$L(n, n, x) = \|x\|$$

and

$$L(m, n, x) - aL(m-1, n, x) \geq \| \mathcal{A}(m, n)x \| \quad (14)$$

for all $(m, n, x) \in \Delta \times X$, with $m > n$.

Theorem 7. *The linear discrete-time system (\mathfrak{A}) is NPIS if and only if there exist a Lyapunov sequence and a nondecreasing sequence $\beta : \mathbb{N} \rightarrow [1, \infty)$ such that*

$$L(m, n, x) \leq \beta(m) \| \mathcal{A}(m, n)x \|, \quad (15)$$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity. Let $d > 1$. We define $L : \Delta \times X \rightarrow \mathbb{R}_+$ by

$$L(m, n, x) = \sum_{k=n}^m d^{m-k} \| \mathcal{A}(k, n)x \|,$$

for all $(m, n, x) \in \Delta \times X$.

First, we observe that

$$\begin{aligned} L(m, n, x) &= \sum_{k=n}^m d^{m-k} \| \mathcal{A}(k, n)x \| \\ &= d^{m-n} \|x\| + \dots + \| \mathcal{A}(m, n)x \| \\ &= dL(m-1, n, x) + \| \mathcal{A}(m, n)x \|, \end{aligned}$$

for all $(m, n, x) \in \Delta \times X$, with $m > n$. Hence

$$L(m, n, x) - aL(m-1, n, x) \geq \| \mathcal{A}(m, n)x \|$$

for every $a \in (1, d)$ and $(m, n, x) \in \Delta \times X$, with $m > n$.

On the other hand, for $d \in \left(1, \frac{1}{r}\right)$ we have that

$$\begin{aligned}
L(m, n, x) &\leq \sum_{k=n}^m N(m) d^{m-k} r^{m-k} \| \mathcal{A}(m, n)x \| \\
&= N(m) \| \mathcal{A}(m, n)x \| \sum_{k=n}^m (dr)^{m-k} \\
&\leq \frac{N(m)}{1-dr} \| \mathcal{A}(m, n)x \| \\
&= \beta(m) \| \mathcal{A}(m, n)x \|.
\end{aligned}$$

Sufficiency. According to (14) we have that

$$\begin{aligned}
L(m, n, x) - aL(m-1, n, x) &\geq \| \mathcal{A}(m, n)x \| \\
L(m-1, n, x) - aL(m-2, n, x) &\geq \| \mathcal{A}(m-1, n)x \| \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \\
L(n+1, n, x) - aL(n, n, x) &\geq \| \mathcal{A}(n+1, n)x \|
\end{aligned}$$

which implies

$$\sum_{j=n}^m a^{m-j} \| \mathcal{A}(j, n)x \| \leq L(m, n, x) \leq \beta(m) \| \mathcal{A}(m, n)x \| \quad (16)$$

But

$$\sum_{j=n}^m a^{m-j} \| \mathcal{A}(j, n)x \| \geq a^{m-n} \| x \|. \quad (17)$$

Now, using (16), (17) and Proposition 1 we obtain that system (\mathfrak{A}) is NPIS. \square

As a consequence of the previous theorem we obtain

Corollary 8. *The linear discrete-time system (\mathfrak{A}) is NPIS if and only if there exist a nondecreasing sequence $\theta : \mathbb{N} \rightarrow [1, \infty)$ such that*

$$\sum_{k=n}^m \| \mathcal{A}(k, n)x \| \leq \theta(m) \| \mathcal{A}(m, n)x \|,$$

for all $(m, n, x) \in \Delta \times X$.

Proof. It results from Definition 2 and the proof of Theorem 7 and Proposition 1. \square

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References

- [1] *Jeffrey J. DaCunha*, Instability results for slowly time varying linear dynamic systems on time scales, J. Math. Anal. Appl. 328 (2007), 1278-1289.
- [2] *W. Hahn*, Stability of Motion, Grundlehren Math. Wiss., vol. 138, Springer, 1967.
- [3] *A. Halanay, D. Wexler*, Qualitative Theory of Impulsive Systems [English translation], Edit. Acad. R.P.R., Bucureşti, 1968.
- [4] *Morris W. Hirsch*, The dynamical systems approach to differential equations, Bull. Amer. Math. Soc. (N.S.), Vol. 11, No. 1(1984), 1-64.
- [5] *J. LaSalle, S. Lefschetz*, Stability by Liapunov's Direct Method, with Applications, Math. Sci. Eng., vol. 4, Academic Press, 1961.
- [6] *A. Lyapunov*, The General Problem of the Stability of Motion, Taylor & Francis, 1992.
- [7] *A.A. Martynyuk*, Stability Analysis of Discrete Systems, Internat. Appl. Mech., Vol. 36, No.7(2000), 835-865.
- [8] *M. Megan, A. L. Sasu and B. Sasu*, Nonuniform exponential unstability of evolution operators in Banach spaces, Glas. Mat. Vol. 36(56)(2001), 287-295.
- [9] *M. Megan, A. L. Sasu and B. Sasu*, Perron Conditions for Uniform Exponential Expansiveness of Linear Skew-Product Flows, Monatsh. Math., Vol. 138 (2003), 145157.

- [10] *N. van Minh, F. Răbiger, and R. Schnaubelt*, Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line, *Integral Equations Operator Theory*, vol. 32, no. 3(1998), 332-353.
- [11] *R. Naulin, C. J. Vanegas*, Instability of discrete systems, *Electron. J. Differential Equations*, Vol. 33 (1998), 1-11.
- [12] *I.-L. Popa, T. Ceașu, M. Megan*, On exponential stability for linear discrete-time systems in Banach spaces, *Comp. Math. Appl.*, Vol. 63 (2012), 1497-1503.
- [13] *I.-L. Popa*, A Note on Power Instability of Linear Discrete-time Systems in Banach Spaces, *An. Univ. Vest Timi. Ser. Mat.-Inform.*, Vol. 1 (2012), 83-89.
- [14] *Wilson J. Rugh*, *Linear system theory*, Prentice-Hall, 1996
- [15] *A.L. Sasu*, Exponential instability and complete admissibility for semigroups in Banach spaces, *Rend. Sem. Mat. Univ. Pol. Torino* - Vol. 63, 2 (2005), 141-151.
- [16] *V.E. Slyusarchuk*, On the instability of difference equations with respect to the first approximation, *Differents. Uravn.*, 22, No.4 (1986), 722-723.
- [17] *B.-G. Wang, Z.-C. Wang*, Exponential dichotomy and admissibility of linearized skew-product semiflows defined on a compact positively invariant subset of semiflows, *Nonlinear Anal. Real World Appl.*, Vol. 10 (2009) 2062-2071.